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RECENT DEVELOPMENTS ON SPIRO'S ADDITIVE UNIQUENESS PROBLEM

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요약. In this article, a brief history and developments on additive uniqueness sets for arithmetic functions are introduced. Let S be a set of arithmetic functions. A set $E \subset \mathbb{N}$ is called an additive uniqueness set for S if $f \in S$ and the condition

$$f(a+b) = f(a) + f(b) \text{ for all } a, b \in E$$

determines f uniquely. Various sets have been shown to be additive uniqueness sets and many other variations of the problem were investigated.

1. Introduction

In 1992 Claudia Spiro [39] proposed an interesting problem. She asked which set E determines an arithmetic function $f : \mathbb{N} \to \mathbb{C}$ uniquely in some set S of arithmetic functions under the condition

$$f(a+b) = f(a) + f(b) \text{ for all } a, b \in E$$

and she called E an *additive uniqueness* set for S. For brief, we abbreviate additive uniqueness (set) to AU.

In the paper she proved that if a multiplicative function f satisfies the condition $f(p+q) = f(p) + f(q)$ for all primes p and q and there exists a prime p_0 such that $f(p_0) \neq 0$, then f is the identity function, where *multiplicative* means an arithmetic function f satisfies $f(1) = 1$ and $f(ab) = f(a)\,f(b)$ if $\gcd(a, b) = 1$. That is, she showed that the set of primes is an AU set for multiplicative functions which do not vanish at some prime numbers.

Since her intriguing result, many mathematicians have been solving enormously various problems related to Spiro's paper. For example, Fang [9] extended Spiro's result to $f(p + q + r) = f(p) + f(q) + f(r)$ for all primes p, q, and r. It was also extended to

$$f(p_1 + p_2 + \cdots + p_k) = f(p_1) + f(p_2) + \cdots + f(p_k)$$

for all primes p_i by Dubickas and Šarka [7]. Their results can be rephrase with the term *k-additive uniqueness*, briefly k-AU, set. Thus, as for the set of prime numbers, Spiro proved 2-AU, Fang proved 3-AU, and Dubickas and Šarka generally proved k-AU.

One can consider various additive uniqueness sets. In 1999 Chung and Phong [4] proved that the set of triangular numbers and the set of tetrahedral numbers are both AU for multiplicative functions.

Another example is the set of squares. In 1996 Chung [6] showed that the set of positive squares is an additive uniqueness set for completely multiplicative functions, but it is not for multiplicative functions. A *completely multiplicative function* is a function f such that $f(ab) = f(a)\,f(b)$ for every $a, b \in \mathbb{N}$. He classified all multiplicative functions which are 2-additive on positive squares. In 2018 Park [26] proved that the set of positive squares is a k-additive uniqueness set for multiplicative functions if $k \geq 3$.

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Other polygonal numbers also have been studied. But, for this case we should consider two kinds of polygonal numbers: ordinary polygonal numbers and generalized polygonal numbers. In general, the study for ordinary polygonal numbers is much more difficult than generalized ones. So most papers are about generalized polygonal numbers.

We would call such problems to find arithmetic (esp. multiplicative) functions with some additive conditions *Spiro problem*.

The additive condition for Spiro problem diverges in various directions. If we change the k-AU condition for squares to

$$f(a_1^2 + a_2^2 + \cdots + a_k^2) = f(a_1)^2 + f(a_2)^2 + \cdots + f(a_k)^2,$$

then f is the identity function when $k \geq 3$ [25]. We can think of this problem as a functional equation to find multiplicative functions commutable with a specific quadratic form. Mathematicians have studied similar problems for various quadratic or cubic forms.

Phong [35] also studied the condition

$$f(p + q + pq) = f(p) + f(q) + f(pq)$$

for primes p and q. This equation is called Hosszú type.

Recently, another related problem was proposed by Pak and Kang [24]. They introduced *parallelogram uniqueness* along with the equation

$$f(a + b) + f(a - b) = 2f(a) + 2f(b)$$

and showed that a multiplicative function satisfying the equation is uniquely determined.

2. k-additive uniqueness

The first non-trivial AU set for multiplicative functions was the set of primes.

Theorem 2.1 (Spiro [39]). *Let f be a multiplicative function and $f(p_0) \neq 0$ for some prime p_0. If f satisfies*

$$f(p + q) = f(p) + f(q)$$

for all primes p and q, then f is uniquely determined to be the identity function.

If the Goldbach conjecture could be proved to be true, this theorem would be obviously true. But, the Goldbach conjecture is still open and it is verified just up to 4×10^{18} [23]. Spiro's idea of bypassing Goldbach conjecture was to use Estermann's result that almost every even positive integer is expressible as the sum of two primes [8].

In order to derive a contradiction from the assumption of $f(n) \neq n$ for some n, she constructed a set H, we would call it *Spiro set*, and calculated the density of a certain set H_n related to H and n. Then, every element in H_n of positive density cannot be a sum of two primes. This clever strategy has been widely used to prove similar problems.

Spiro's result was generalized to the k-AU. That is, the following holds.

Theorem 2.2 (Fang [9], Dubickas–Šarka [7]). *Fix $k \geq 3$. If a multiplicative function f satisfies*

$$f(p_1 + p_2 + \ldots + p_k) = f(p_1) + f(p_2) + \ldots + f(p_k)$$

for all primes $p_1, p_2, \ldots, p_k$, then f is the identity function.

Spiro had asked whether the set of sufficiently large primes can be AU for multiplicative functions and it was negatively answered. The counterexample is just the set of odd primes [2]. That is, it is not AU for multiplicative functions. This result was extended as follows:

Theorem 2.3 (Lebowitz-Lockard [21]). *A set S of primes is AU for multiplicative functions f with $f(p_0) \neq 0$ for some prime p_0 if and only if it contains every prime that is not the larger element of a twin prime pair and at least one element of 5, 7.*

So, $S = \{2, 3, 11, 17, 23, 29, \dots\} \cup \{5\}$ and $S = \{2, 3, 11, 17, 23, 29, \dots\} \cup \{7\}$ are two minimal AU sets for multiplicative functions.

Another AU set for multiplicative is the set of triangular numbers. In addition, the set of tetrahedral numbers is also AU for multiplicative functions.

Theorem 2.4 (Chung–Phong [4]). *Let $H_k = \left\{ \frac{n(n+1)\dots(n+k-1)}{1\cdot 2 \dots k} \mid n \in \mathbb{N} \right\}$. Then, H_2 and H_3 are AU for multiplicative functions.*

Chung and Phong also conjectured that H_k is AU for $k \geq 4$, which is still open. Also, the set H_2 of triangular numbers is k-AU for all $k \geq 2$ [27].

Not all polygonal numbers are AU. Chung [6] showed that the set of squares is not AU. He classified all multiplicative functions f satisfying the condition $f(m^2 + n^2) = f(m^2) + f(n^2)$. This result was one of the first studies after Spiro's paper. But, curiously, the set of squares is k-AU for all $k \geq 3$ [26].

Theorem 2.5 (Chung [6], Park [26]). *The set of nonzero squares is not 2-AU for multiplicative functions, but is k-AU for all $k \geq 3$.*

As a variant of Theorem 2.5, Phong [36] classified multiplicative functions f satisfying

$$f(m^2 + k + n^2 + 1) = f(m^2 + k) + f(n^2 + 1)$$

for all $m, n \in \mathbb{N}$ and fixed $k \in \mathbb{N}$ according to $f(2)$ and $f(5)$. Park [29] showed that the set

$$\{m^2 + n^2 \mid m, n \in \mathbb{N}\} = \{2, 5, 8, 10, 13, 18, 17, 20, 25, \dots\}$$

of sums of two positive squares is AU if $f(3)\,f(11) \neq 0$. Hasanalizade [11] showed that the set

$$\left\{ \sum_{i=n}^{n+k} i^2 \mid n, k \in \mathbb{Z}, n \geq 1, k \geq 0 \right\} = \{1, 4, 5, 9, 13, 14, 16, 25, 29, \dots\}$$

of sums of consecutive squares is k-AU.

The next target for AU of polygonal numbers is the pentagonal numbers $\frac{n(3n-1)}{2}$. But, in this case, we should consider the ordinary pentagonal numbers and the generalized pentagonal numbers. Note that the generalized hexagonal number is same as triangular numbers.

It is easily shown that the set of generalized pentagonal numbers is AU [19]. Hasanalizade [10] proved k-AU of generalized pentagonal numbers for $k \geq 3$. His proof for 3-AU needed to assume GRH (Generalized Riemann Hypothesis), but a new proof without GRH was later published [31].

Theorem 2.6 (Kim–Kim–Lee–Park [19], Hasanalizade [10], Park [31]). *The set*

$$\left\{ \frac{n(3n-1)}{2} \mid n \in \mathbb{Z}, n \neq 0 \right\} = \{1, 2, 5, 7, 12, 15, \dots\}$$

of generalized pentagonal numbers is k-AU for all $k \geq 2$.

But, The proof for AU of ordinary pentagonal numbers, $1, 5, 12, 22, 35, \dots$, is complicated than that of generalized pentagonal numbers [20]. The similar method could be used to prove 2-AU of ordinary hexagonal numbers. Also, it is extended to k-AU [13].

For ordinary hexagonal numbers the above theorem could be extended.

Theorem 2.7 (Kim–Kim–Lee–Park [20], Hasanalizade–Park–Inochkin [13]). *The set*

$$\{n(2n-1) \mid n \in \mathbb{Z}, n > 0\} = \{1, 6, 15, 28, 45, 66, \dots\}$$

of ordinary hexagonal numbers is k-AU with $k \geq 2$.

Not much is known about other polygons. The set of generalized octagonal numbers and the set of generalized nonagonal numbers were investigated.

Theorem 2.8 (Hasanalizade–Park [12]). *The set*

$$\{n(3n-2) \mid n \in \mathbb{Z}, n \neq 0\} = \{1, 5, 8, 16, 21, 33, \dots\}$$

of generalized octagonal numbers is k-AU for all $k \geq 4$.

Theorem 2.9 (Park [34]). *The set*

$$\left\{\frac{n(7n-5)}{2} \mid n \in \mathbb{Z}, n \neq 0\right\} = \{1, 6, 9, 19, 24, 39, \dots\}$$

of generalized nonagonal numbers is 2-AU.

The next AU candidates for generalized polygonal numbers are 14 and 17.

Here we summarize the AU properties of polygonal numbers:

polygonal	AU	non-AU	papers
triangular	k-AU with $k \geq 2$	-	[4], [27]
square	k-AU with $k \geq 3$	non-2-AU	[6], [26]
pentagonal	2-AU	unknown	[20]
hexagonal	k-AU with $k \geq 2$	-	[20], [13]
gen'd pentagonal	k-AU with $k \geq 2$	-	[19], [10], [31]
gen'd octagonal	k-AU with $k \geq 4$	non-$(2,3)$-AU	[12]
gen'd nonagonal	2-AU	unknown	[34]

The minimal sets of each polygonal numbers for multiplicative functions are not yet studied.

Another generalization of polygonal numbers is polytope numbers. Like the AU of tetrahedral numbers H_3 in Theorem 2.4, we can study the AU property of positive cubic numbers.

Theorem 2.10 (Park [32]). *The set of positive cubic numbers is not 2-AU, but is k-AU for $k \geq 3$.*

Other polytope numbers, for example octahedral numbers, are not yet studied.

3. Variants of functional equations

Instead of the equation $f(m^2+n^2) = f(m^2)+f(n^2)$, we can consider the slightly modified equation

$$f(m^2+n^2) = f(m)^2 + f(n)^2.$$

This can be regarded as the commutable relation $f(Q(m,n)) = Q(f(m), f(n))$ for the quadratic form $Q(x,y) = x^2 + y^2$.

This problem was studied by Bašić. Similar to Chung's result, the multiplicative functions satisfying the relation are not determined uniquely. But, if we extend variables, the multiplicative function is determined to be the identity function.

Theorem 3.1 (Bašić [1], Park [25]). *Let a multiplicative function f satisfy*

$$f(a_1^2+\ldots+a_k^2)=f(a_1)^2+\ldots+f(a_k)^2$$

for all positive integers a_i. If $k=2$, then f is not uniquely determined. But, if $k\geq 3$, then f is the identity function.

The similar property also holds for positive cubic numbers [32].

Some mathematicians studied the same problem for different quadratic forms $Q(x,y)$.

Theorem 3.2 (Khanh [17]). *Let $D\in\mathbb{N}$. If an arithmetic function f satisfies*

$$f(x^2+Dy^2)=f(x)^2+Df(y)^2,$$

then $f(n)=0$, $f(n)=\pm n$, or $f(n)=\pm\frac{1}{D+1}$.

Theorem 3.3 (Park [30]). *If a multiplicative function f satisfies*

$$f(x^2+xy+y^2)=f(x)^2+f(x)f(y)+f(y)^2$$

for all $x,y\in\mathbb{N}$, then f is the identity function. But, if f satisfies

$$f(x^2-xy+y^2)=f(x)^2-f(x)f(y)+f(y)^2$$

for all $x,y\in\mathbb{N}$, then f is one of three kinds of functions.

Phong and Szeidl [37] generalized the first result of Theorem 3.2 to arithmetic functions satisfying $f(n^2+nm+m^2)=f(n)^2+f(n)f(m)+f(m)^2$ and Khanh [18] found solutions of $f(n^2+Dnm+m^2)=f(n)^2+Df(n)f(m)+f(m)^2$ for $D=2,3$.

Phong and Szeidl [38] also generalized the second result of Theorem 3.2 to arithmetic functions f satisfying $f(n^2-Dnm+m^2)=f(n)^2-Df(n)f(m)+f(m)^2$ with $D=1,2$.

Spiro's original problem for primes was also studied in the other ways.

Theorem 3.4 (Chen–Fang–Yuan–Zheng [3]). *Let f be a multiplicative function such that there exists a prime p_0 at which f does not vanish and let n_0 be a fixed integer with $1\leq n_0\leq 10^6$. If*

$$f(p+q+n_0)=f(p)+f(q)+f(n_0)$$

holds for all primes p and q, then f is the identity function.

If Goldbach conjecture is true, the restriction for n_0 can be removed. This variant can be considered for negative n_0. But, $n_0<-3$ is excluded for f is an arithmetic function.

Theorem 3.5 (Park [28]). *Let $1\leq n_0\leq 3$. Suppose the condition for f as same as Theorem 3.4. If f satisfies*

$$f(p+q-n_0)=f(p)+f(q)-f(n_0)$$

for all primes p and q, then f is the identity function or a constant function $f(n)=1$.

For $n_0=2$, one more function is possible.

Functional equations for specific sets were also investigated. The equation of Theorem 3.6 is an example, which is called Hosszú type.

Theorem 3.6 (Phong [36]). *If a completely multiplicative function f satisfies*

$$f(p+q+pq)=f(p)+f(q)+f(pq)$$

for all primes p and q and $f(p_0)\neq 0$ for some prime number p_0, then f is the identity function.

There are plenty of problems to classify arithmetic functions f and g satisfying

$$f(a+b) = g(a) + g(b)$$

for all a and b in a specific set.

Theorem 3.7 (Kátai–Phong [14]). *Let $k \geq 3$ and $M_k = \{p_1+p_2+\cdots+p_k \,|\, p_i : \text{ prime}\}$. If arithmetic functions f and g satisfy*

$$f(p_1 + p_2 + \cdots + p_k) = g(p_1) + g(p_2) + \cdots + g(p_k)$$

for all primes p_i, then there are two constant A and B such that $f(n) = An + kB$ for all $n \in M_k$ and $g(p) = Ap + B$ for all prime p.

They conjecture that Theorem 3.7 holds for $k = 2$. See [15, 16] for their other related works.

Recently, Pak and Kang suggested a new type of problem. They call it parallelogram uniqueness.

Theorem 3.8 (Pak–Kang [24]). *Let f be a multiplicative function and $f(0) = 0$. If the Goldbach conjecture is true and f satisfies*

$$f(p+q) + f(p-q) = 2f(p) + 2f(q)$$

for all primes p and q ($p \geq q$), then $f(n) = n^2$.

The proof of Theorem 3.8 without Goldbach conjecture is not yet found. There are only a few sets that have been studied, as there are not enough studies yet.

Theorem 3.9 (Park [33]). *If a multiplicative function f satisfies*

$$f(a^2+b^2) + f(a^2-b^2) = 2f(a^2) + 2f(b^2)$$

for all positive integers $a > b$ and $f(4) \neq 0$, then $f(n) = n^2$.

4. Open problems

Many unsolved problems have been proposed since Spiro's paper. Here we list a few unsolved problems.

4.1. Fibonacci numbers. The first is Spiro's own open problem. She asked whether the set of Fibonacci numbers is AU or not [39]. It has been unsolved for 30 years. It can be shown just that $f(2^r) = 2^r$ [22]. There is little progress on this problem.

4.2. High dimensional polytope numbers. Chung–Phong's conjecture about H_k for $k \geq 4$ is also unsolved for over 20 years [4]. Similarly, we can consider the set of n^r for fixed r. For $r = 2, 3$ the sets are k-AU for $k \geq 3$ [26, 32].

4.3. Polygonal numbers. Which sets of (generalized) polygonal numbers are AU? We have each set of s-polygonal numbers are AU for $s = 3, 5, 6, 9$ [4, 20, 27, 34].

4.4. Sequential k-AU. For many investigated sets, if it is k-AU, then it is also ℓ-AU with $k \leq \ell$. For example, the set of positive squares is k-AU for $k \geq 3$. Is there a set which is k-AU but is not $(k+1)$-AU?

4.5. Almost k-AU. The set of generalized octagonal numbers is not 2-, 3-AU and is k-AU for $k \geq 4$ [12]. If $f(a+b+c) = f(a)+f(b)+f(c)$ for all generalized octagonal numbers, then $f(n) = n$ for $n \neq 4$. We could think of this set as *almost* 3-AU with only one exception. So far, this set is the only example with this property.

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