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GEOMETRIC CONSTRUCTIONS VIA ORIGAMI: FROM CLASSICAL METHODS TO MULTIPLE FOLD TECHNIQUES

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Abstract. We examine geometric constructions using Origami (paper folding), exploring the progression from classical ruler-and-compass methods to advanced multiple fold Origami techniques. We begin by reviewing ruler-and-compass constructions and their algebraic characterization through field extensions. We then introduce the seven Huzita-Justin axioms for single-fold Origami and demonstrate that axioms O_1 through O_5 produce exactly the ruler-and-compass constructible points, while axiom O_6 extends constructibility to solutions of cubic equations, including angle trisection.

We investigate multiple fold axioms and present the result that $(n-2)$-fold Origami can construct real roots of degree n polynomials using Lill's method. Our main contribution is proving that all quadratic equations with complex coefficients can be solved using two-fold Origami. We introduce Lill's Method for Complex Roots (LMCR), which finds polynomial roots through sequences of similar triangles in the complex plane, and provide an explicit construction method via the axiom AL13a7bb. This work demonstrates how Origami constructions transcend classical geometric limitations.

1. Introduction

The classical method of geometric construction uses only a straightedge and a compass. With these tools, one can draw a line through two given points and construct a circle with a given center and radius equal to the distance between any two given points, as well as determine their intersections. It is well known that such constructions produce exactly the numbers lying in a tower of quadratic field extensions over $\mathbb{Q}$. Because of this algebraic restriction, certain problems, such as trisecting an arbitrary angle or solving a general cubic equation, cannot be solved by ruler-and-compass constructions alone. Origami constructions, however, provide a powerful alternative that overcomes these limitations. Standard references such as [7] and [9] provide detailed introductions to the mathematical structure underlying paper-folding constructions.

A formal axiomatic description of single-fold Origami is given by the Huzita–Justin axioms. Using these axioms, one can trisect an arbitrary angle and solve all quadratic, cubic, and quartic equations with rational coefficients [12]. That is, single-fold Origami can construct any real number lying in a tower of quadratic and cubic field extensions over $\mathbb{Q}$.

Alperin and Lang [2] extended these axioms to n-fold Origami, which allow two or more fold lines to be created simultaneously under mutual alignment constraints. Combining Lill's method with the n-fold axioms, they showed that every real root of a polynomial of

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degree n can be constructed using $(n-2)$-fold Origami. However, this bound represents a worst-case requirement. In practice, many polynomials can be solved with fewer folds. In 2004, Lang proved that angle quintisection is possible using two-fold Origami [11]. Building on this result, Nishimura [15] proved that two-fold Origami can construct roots of any quintic polynomial, which also enables angle quintisection, and König and Nedrenco [10] further showed that two-fold Origami can also solve septic equations.

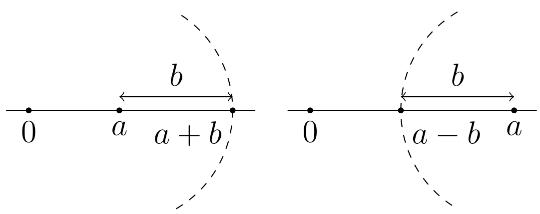
In this paper, we present a comprehensive exposition of these Origami construction methods. We provide detailed proofs of the Huzita-Justin axioms, demonstrate constructions for angle trisection and solutions of cubic equations, and examine the theory of multiple-fold Origami. We also introduce an extension of Lill's method applicable to complex roots and give an explicit two-fold Origami construction for quadratic equations with complex coefficients.

2. Euclidean constructions - Ruler and Compass

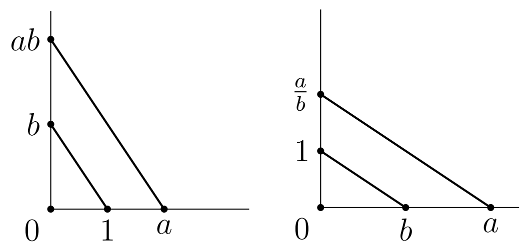
In the classical geometric construction, there are two permitted operations on the points in the Euclidean plane. We start with $P = P_0$, a set of points in the plane.

- (C_1) Given two distinct points p_1 and p_2 in P, draw a line that passes through p_1 and p_2 with a ruler.
- (C_2) Given two distinct points p_1 and p_2 in P, draw a circle centered at p_1 with radius $|p_1 - p_2|$ with a compass.

Let P_1 be the set of points in P_0 and intersection points of lines and circles obtained by means of the operations C_1 and C_2. Performing C_1 and C_2 on $P = P_1$, we obtain the new set P_2. We can continue this process and call the points obtained in this process *RC-constructible from* P_0. Moreover, we call a point that is constructible from $\{(0,0),(1,0)\}$ *RC-constructible*, and we say that the coordinates of such points are *RC-constructible numbers*. Important feature of RC-construction includes constructing the perpendicular bisector of a line segment, bisecting an angle, drawing a perpendicular line from a point to a line, etc. Moreover, it is easy to see that the set of RC-constructible numbers is closed under the four arithmetic operations, forming a field (see Fig. 1). This implies that the field of RC-constructible numbers contains the rationals $\mathbb{Q}$.



(a) Addition: $a+b$ (b) Subtraction: $a-b$



(c) Multiplication: $a \times b$ (d) Division: $\frac{a}{b}$

FIGURE 1. RC-constructible numbers are closed under arithmetic operations

When constructing new points, we add intersection points obtained through the operations C_1 or C_2. There are three types of intersections: line with line, line with circle, and circle with circle. Solving the corresponding equations in each case leads to a quadratic equation. Thus, if p is a constructible number, then $[\mathbb{Q}(p) : \mathbb{Q}]$ must be a power of 2. Moreover, as shown in Fig. 2, we can construct $\sqrt{a}$ for any constructible $a > 0$ using standard ruler-and-compass methods.

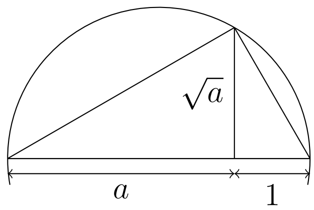


FIGURE 2. Construction of $\sqrt{a}$ using the RC

By the quadratic formula, we obtain the following result.

THEOREM 2.1. *RC-constructions can solve any quadratic equation with RC-constructible coefficients.*

However, arbitrary angle trisection is impossible with RC-constructions. For example, trisecting a 60° angle (equivalently, constructing a 20° angle) requires constructing $\sin 20°$. Using the triple-angle identity,

$$\sin 60° = 3\sin 20° - 4\left(\sin 20°\right)^3,$$

we see that $\sin 20°$ satisfies the cubic equation

$$4x^3 - 3x + \frac{\sqrt{3}}{2} = 0.$$

Since $[\mathbb{Q}(\sin 20°) : \mathbb{Q}] = 6$, which is not a power of 2, $\sin 20°$ cannot be constructed using RC-methods.

In the next section, we introduce Origami constructions, including methods that enable arbitrary angle trisection.

3. Origami constructions - Paper Folding

If we consider a crease formed by Origami to be a line, we may write *Huzita-Justin axioms* as follows:

- (O_1) Given two distinct points p_1 and p_2, there is a unique line that passes through p_1 and p_2.
- (O_2) Given two distinct points p_1 and p_2, there is a unique line that places p_1 onto p_2.
- (O_3) Given two lines L_1 and L_2, there is a line that places L_1 onto L_2.
- (O_4) Given a point p and a line L, there is a unique line perpendicular to L that passes through p.
- (O_5) Given two points p_1 and p_2 and a line L_1, there is a line that places p_1 onto L_1 and passes through p_2.
- (O_6) Given two points p_1 and p_2 and two lines L_1 and L_2, there is a line that places p_1 onto L_1 and p_2 onto L_2.
- (O_7) Given a point p and two lines L_1 and L_2, there is a line that places p onto L_1 and is perpendicular to L_2.

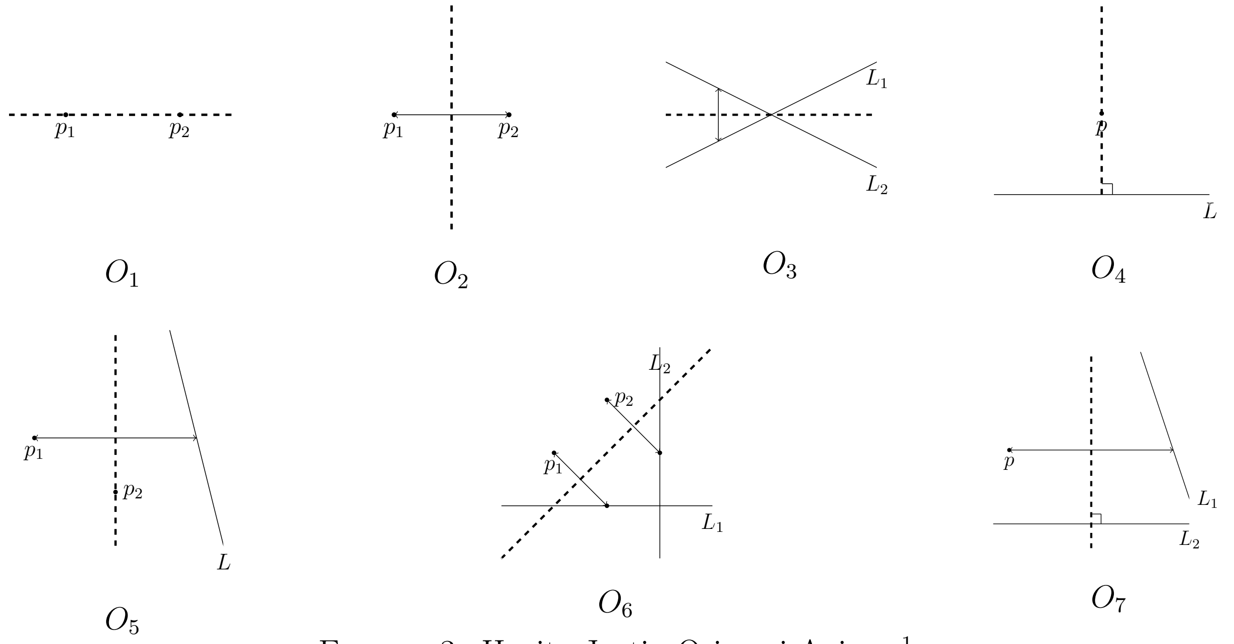


FIGURE 3. Huzita-Justin Origami Axioms[1]

In the context of the Huzita–Justin axioms and the additional Origami axioms introduced later in this paper, the term axiom refers to a basic folding operation rather than a logical axiom in the usual mathematical sense. The formal definition used by Alperin and Lang [2] is the following:

DEFINITION 3.1. A one-fold axiom is a minimal set of alignments that define a single fold line on a finite region of the Euclidean plane with a finite number of solutions.

A line constructed using the above axioms is called an *Origami-constructible line*. Points obtained as intersections of Origami-constructible lines are called *Origami-constructible points*, and each coordinate of such a point is referred to as an *Origami-constructible number*. Note that not all points lying on an Origami-constructible line are necessarily constructible.

It is straightforward to verify that any point constructible using axioms O_1, O_2, O_3 and O_4 can also be obtained by RC-constructions, because O_1 corresponds to C_1, O_2 produces the perpendicular bisector of $\overline{p_1p_2}$, O_3 performs angle bisection between L_1 and L_2 when they intersect, and O_4 constructs a perpendicular line to L_1 through p_1. Including O_5 yields all RC-constructible points, as O_5 is equivalent to finding the intersection of L_1 with the circle centered at p_2 of radius $|p_1 - p_2|$. The resulting fold line is the perpendicular bisector of p_1 and the intersection point if any. It was shown by Auckly and Cleveland [4] that constructions using only O_1 through O_4 are strictly weaker than classical RC-constructions. Later, Alperin [1] proved that constructions using O_1 through O_5 generate exactly the set of RC-constructible points.

To obtain constructible points beyond those achievable by ruler and compass, one must consider axioms O_6 and O_7. In fact, Alperin and Lang [2, Section 2] proved that the Huzita–Justin axioms provide a complete set of axioms for single-fold origami constructions, in the sense that every possible single fold is derivable from these axioms. However, O_7 does not yield any new constructible points. To see why, consider the case when a point p and lines L_1 and L_2 are given (see Fig.4). One may first construct a line L_3 parallel to L_2 passing through point p, then find its intersection with L_1, say q. Folding p onto q produces

[1]Interactive visualizations of these axioms are available at this link (See 1).

the same fold line described by O_7, meaning that this axiom yields nothing beyond O_1 through O_6.

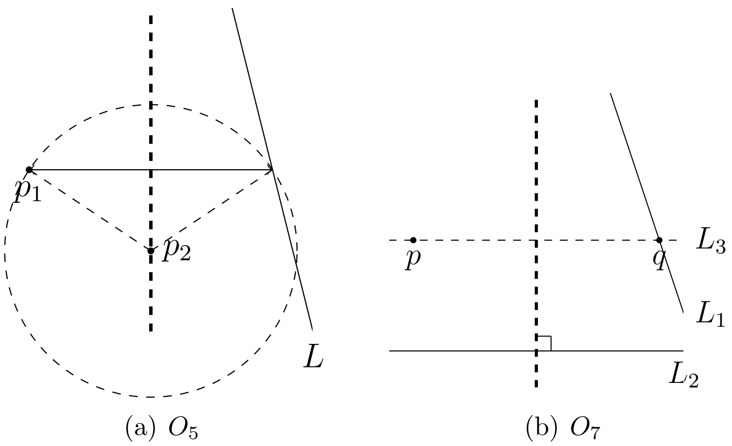


FIGURE 4. Drawing the Axioms O_5 and O_7 by using ruler and compass

We next observe that axiom O_6 is sufficient to solve cubic equations. Before proving this fact, we illustrate how O_6 enables angle trisection.

As shown in Fig.5, let p_1 be the lower left corner of the square Origami paper, and let θ denote the angle between the bottom edge and the line L_1 at p_1. Choose an arbitrary point p_2 on the left edge of the square.

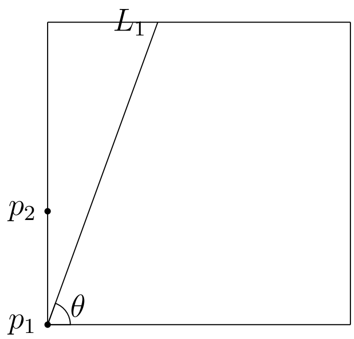


FIGURE 5. Trisection 1[2]

First, apply axiom O_2 to fold p_1 onto p_2, and call the resulting fold line F_1. Then apply O_6 to fold p_1 onto F_1 (mapping it to p_3) and to fold p_2 onto L_1 (mapping it to p_4), as shown in Fig.6. Here F_2 denotes the resulting fold line.

[2]Interactive visualizations of the angle trisection process is available at this link (See 2).

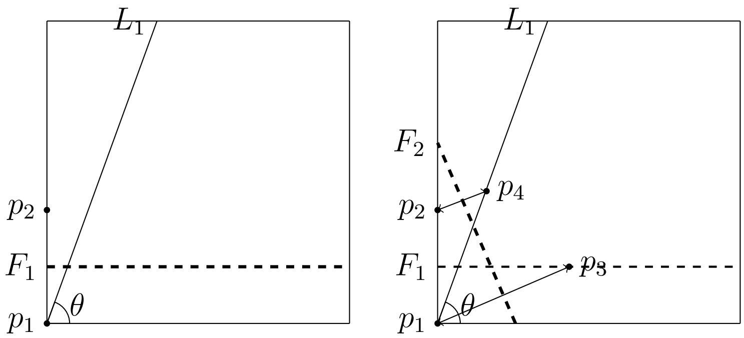


FIGURE 6. Trisection 2

Finally, apply axiom O_3 to fold the extension of the segment $\overline{p_1p_3}$, denoted L_2, onto L_1. The final fold F_3 bisects $\angle p_3p_1p_4$, thereby producing a trisection of the original angle θ.

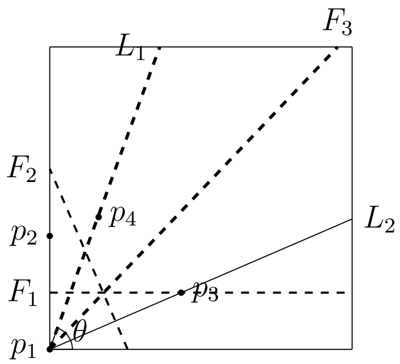


FIGURE 7. Trisection 3

To verify that we have really trisected the angle, let p_5 and p_6 be the intersections of F_2 with F_1 and the bottom edge of the paper, respectively.

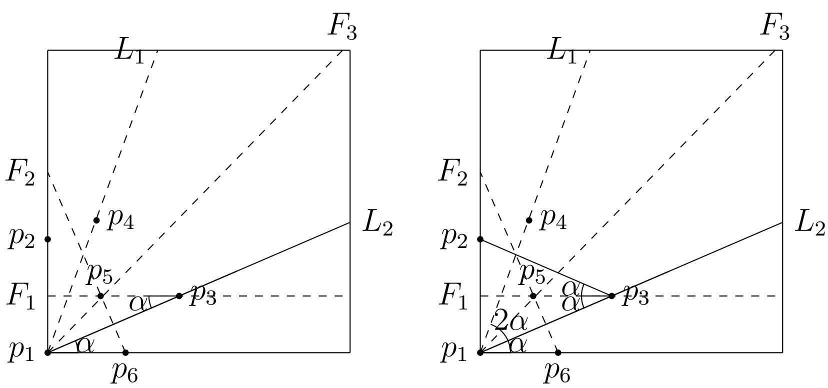


FIGURE 8. Trisection 4

Since $\overline{p_1p_6} \parallel \overline{p_3p_5}$ are both parallel to F_1, the angles $\angle p_6p_1p_3$ and $\angle p_1p_3p_5$ are equal, and we denote their common measure by α. Next, because F_2 is the perpendicular bisector of both $\overline{p_2p_4}$ and $\overline{p_1p_3}$, we have $\angle p_2p_3p_1 = \angle p_4p_1p_3$ and $\angle p_5p_1p_3 = \angle p_5p_3p_1 = \alpha$. Moreover, since F_1 is the perpendicular bisector of $\overline{p_1p_2}$, it follows that $\angle p_2p_3p_5 = \angle p_1p_3p_5 = \alpha$.

Therefore the angle $\angle p_4p_1p_3$ must have measure 2α, and hence $\alpha = \frac{\theta}{3}$. This confirms that the construction indeed produces a trisection of the angle.

In fact, using the Huzita-Justin axioms, one can solve all quadratic, cubic, and quartic equations with rational coefficients. We now give a proof of solving a cubic equation with Origami.

THEOREM 3.2. *Real roots of any cubic polynomial with rational coefficients can be constructed by Origami.*

In fact, the theorem holds for any cubic polynomial with Origami constructible coefficients as seen in the proof below. Before proving the theorem, we review some basic properties of Origami. If one folds the point p onto a point on the line L, say q, the fold line is the perpendicular bisector of $\overline{pq}$. Thus every point on the fold has equal distance to p and to L. Folding p onto all points of L generates a parabola with the focus p and the directrix L, and the fold line in each such fold is the tangent line to the parabola.

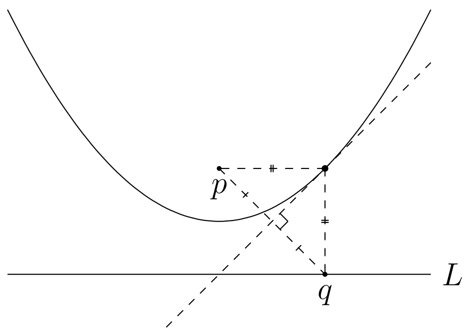


FIGURE 9. O_2 with a point on a line

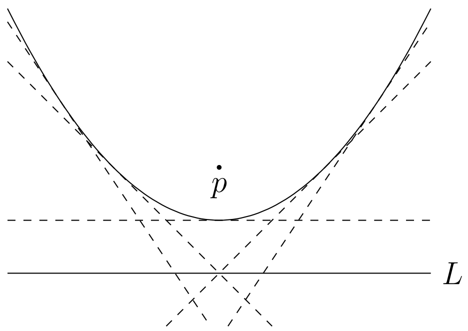


FIGURE 10. Parabola and fold lines[3]

Hence, axiom O_6 is equivalent to finding a common tangent to two parabolas.

[3]Interactive visualization showing how fold lines (tangent lines) form the parabola envelope: link (See 3).

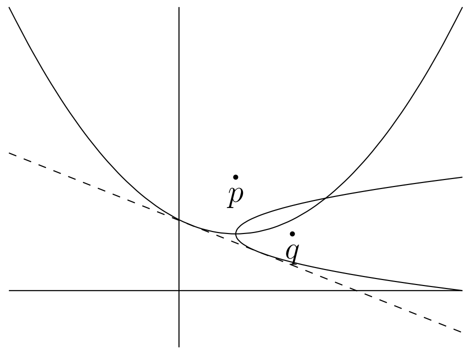


FIGURE 11. O_6 and simultaneous tangent line

Proof of Theorem 3.2. We follow the argument in [5, p.275]. For arbitrary Origami constructible numbers a and nonzero b, consider two parabolas

$$C_a : \left(y - \frac{1}{2}a\right)^2 = 2bx \text{ and } C_b : y = \frac{1}{2}x^2.$$

We will show that the slope m of any common tangent line to both parabolas satisfies $m^3 + am + b = 0$. We exclude the case $b = 0$, because the equation then reduces to $m^3 + am = m(m^2 + a) = 0$, and thus does not represent a nontrivial cubic equation. Because $b \neq 0$, the derivative of C_a is

$$\frac{dy}{dx} = \frac{b}{y - \frac{a}{2}},$$

which is never zero. Hence any common tangent must have nonzero slope, so we restrict to $m \neq 0$.

Let T be a common tangent line to parabolas C_a and C_b at (x_1, y_1) and (x_2, y_2), respectively. If m is the slope of T, then

$$(x_1, y_1) = \left(\frac{b}{2m^2}, \frac{b}{m} + \frac{a}{2}\right) \quad \text{and} \quad (x_2, y_2) = \left(m, \frac{m^2}{2}\right).$$

Hence

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\frac{m^2}{2} - \left(\frac{b}{m} + \frac{a}{2}\right)}{m - \frac{b}{2m^2}} = \frac{m^4 - 2bm - am^2}{2m^3 - b},$$

which implies that

$$m = 0 \quad \text{or} \quad m^3 + am + b = 0.$$

Since $m \neq 0$, we have $m^3 + am + b = 0$. The fold line constructed by Origami has the form $F : y = mx + c$ for some $c \in \mathbb{R}$. The slope m is determined geometrically by folding. To construct the value m as a number, one may intersect F with the x-axis to obtain $\left(-\frac{c}{m}, 0\right)$, and then intersect F with the vertical line $x = -\frac{c}{m} + 1$ to obtain the point $\left(-\frac{c}{m} + 1, m\right)$, whose y-coordinate yields m.

Any cubic equation $x^3 + a'x^2 + b'x + c' = 0$ with Origami-constructible coefficients can be reduced to the depressed form $y^3 + ay + b = 0$ by the substitution $y = x - \frac{a'}{3}$. Thus, one real root can be constructed using axiom O_6. Since Origami constructions include all RC-constructions, the remaining quadratic factor may also be solved using Origami if it has real roots. Therefore, all real roots of any cubic equation with Origami-constructible coefficients can be constructed. This includes, as a special case, all cubics with rational coefficients.

□

4. Multiple Folding to Solve Higher-Degree Polynomial Equations

As shown in the previous section, single-fold Origami allows angle trisection and the construction of all real roots of cubic equations. It has also been shown that single-fold Origami can solve quartic equations [6]. However, it cannot, in general, quintisect an angle or solve a general quintic equation. To extend Origami methods to higher-degree equations, Alperin and Lang [2] used multiple folds, in which two or more fold lines are created simultaneously under prescribed alignment constraints.

Before discussing multiple folds, we restate the classical axiom O_6 using reflection notation. Let $F(P)$ denotes the reflection of a point P across a (fold) line F and $F(L)$ denotes the reflection of a line L. Then axiom O_6 specifies a fold line F satisfying:

$$\begin{cases} F(p_1) \in L_1, \\ F(p_2) \in L_2. \end{cases}$$

Two-fold axioms extend this idea by requiring two fold lines to be constructed simultaneously under mutual constraints. Each fold may depend on the other, and the alignment conditions must hold at the same time. Alperin and Lang identified ten fundamental alignment types for two-fold constructions [2, p.11], and showed that combinations of these alignments yield 489 distinct two-fold axioms [2, p.13, 14]. For example, given two points p_1, p_2 and three lines L_1, L_2 and L_3, a two-fold axiom may require constructing two fold lines F_1 and F_2 satisfying

$$\begin{cases} F_1(p_1) \in L_1, \\ F_1(L_3) = F_2, \\ F_2(p_2) \in L_2. \end{cases}$$

This axiom, denoted $AL4a6ab$, plays an essential role in constructions used to solve quintic equations [15]. Although some configurations, such as the one shown in Fig.6, appear to involve two fold lines, they are not regarded as two-fold axioms unless both folds must be constructed simultaneously rather than sequentially.

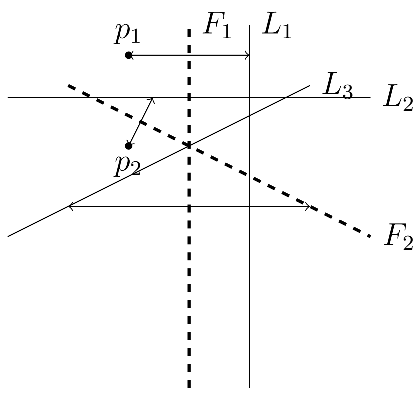


FIGURE 12. Example of Two-fold Axiom called $AL4a6ab$

DEFINITION 4.1. An n-fold axiom is a minimal set of alignments that define n simultaneous fold lines on a finite region of the Euclidean plane with a finite number of solutions.

Alperin and Lang further proved that every real root of a degree n polynomial can be constructed using $(n-2)$-fold Origami via Lill's method [8,13]. Lill's method is a geometric procedure that encodes the coefficients of a polynomial into a piecewise path.

Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial with real coefficients, and assume it has at least one real root. Lill's method represents the polynomial geometrically through a stepwise construction. Begin at the origin O, facing the positive x-axis. For each coefficient a_k, starting from a_n, move a distance $|a_k|$ along the current facing direction: if $a_k > 0$, move forward; if $a_k < 0$, move backward along the same line; and if $a_k = 0$, make no translation but proceed to the next step. After each translation (including the case $a_k = 0$), rotate $90°$ counterclockwise and continue the process until the final coefficient a_0 has been used. The endpoint of this construction is denoted T, and the extended lines supporting each segment are labeled $L_n, L_{n-1}, \ldots, L_0$. If a coefficient is zero, the corresponding segment has length 0, but L_k is still defined as the infinite line through the current point in the current facing direction.

As a simple example, consider the polynomial $f(x) = x^2 - 1$ with coefficients $(1, 0, -1)$. Starting at the origin and facing the positive x-axis, we first move one unit forward for the coefficient 1, then rotate $90°$ counterclockwise. The next coefficient is 0, so no translation occurs, but we again rotate $90°$. At the final coefficient -1, we move one unit backward relative to the current facing direction, which brings us to the point $(2, 0)$. The supporting lines determined by these steps are $L_2 : y = 0$, $L_1 : x = 1$, and $L_0 : y = 0$.

We now describe how Lill's method relates the geometric path to the real roots of $f(x)$. Starting from the origin O, consider a ray that departs from the positive x-axis with angle θ. Let this ray first meet the line L_{n-1}, then reflect by $90°$ toward L_{n-2}, and so on, continuing to reflect orthogonally from

$$L_{n-1}, L_{n-2}, \ldots, L_1, L_0$$

in this order. If, after the final reflection at L_0, the ray passes through the point T (and the path does not pass through any intermediate vertex of the coefficient segments for $i \geq 2$), we obtain a broken ray from O to T as illustrated in Figure 13.

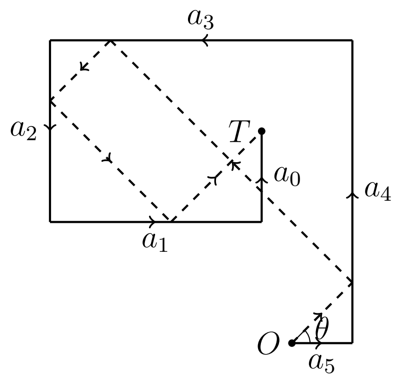


FIGURE 13. Lill diagram for $f(x) = a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$ with all positive coefficients.[4]

We claim that in this case $-\tan\theta$ is a real root of $f(x)$. To see this, we introduce a sequence of signed "remainders" y_k determined by the distances along the coefficient lines. As the ray travels from O to L_{n-1} and along L_{n-1}, the signed distance it covers on L_{n-1} is $a_n \tan\theta$ as shown in Figure.14, so the remaining signed length on that line is

$$y_{n-1} = a_{n-1} - a_n \tan\theta.$$

[4]Interactive visualization of Lill's method for finding polynomial roots geometrically: link (See 4).

When the ray reflects to L_{n-2}, a similar consideration shows that the signed distance along L_{n-2} is $y_{n-1}\tan\theta$, so the remaining length on L_{n-2} is

$$y_{n-2} = a_{n-2} - y_{n-1}\tan\theta.$$

Proceeding in this way, we define recursively for $k = n-1, n-2, \dots, 1$,

$$y_{k-1} = a_{k-1} - y_k \tan\theta.$$

Since the broken ray ends exactly at T, the remaining length on the last line L_0 must be zero, that is,

$$y_0 = 0.$$

By expanding the recurrence, one checks that

$$y_0 = a_0 - a_1\tan\theta + a_2(\tan\theta)^2 - \cdots + (-1)^n a_n(\tan\theta)^n = f(-\tan\theta).$$

Thus the condition $y_0 = 0$ is equivalent to $f(-\tan\theta) = 0$, which proves that $-\tan\theta$ is a real root of $f(x)$ whenever the reflected path with departure angle θ starts at O and ends at T.

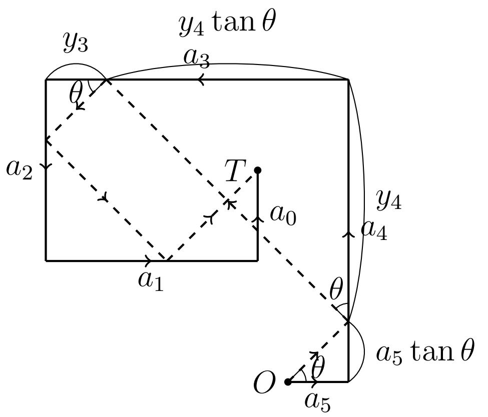


FIGURE 14. Proof of Lill's method

Moreover, we can realize the solution path in Lill's method for a polynomial of degree n by using a single $(n-2)$-fold operation. In the Lill diagram associated with the polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$, the solution ray from O to T consists of n straight segments joined by $n-1$ right-angle turns on the lines $L_{n-1}, \dots, L_1$.

To construct this path by folding, let $F_1, \dots, F_{n-2}$ denote the fold lines corresponding to the interior segments. These fold lines are determined simultaneously by the $(n-2)$-fold axioms under the following alignment conditions (see Fig. 15):

$$\begin{cases} F_i \perp F_{i+1}, & \text{for } i = 1, 2, \dots, n-3, \\ F_i \cap F_{i+1} \in L_{n-i-1}, & \text{for } i = 1, 2, \dots, n-3, \\ F_1(O) \in L_a, & \\ F_{n-2}(T) \in L_b, & \end{cases}$$

where the boundary conditions are encoded by two auxiliary constraint lines: L_a, defined as $x = 2a_n$, and L_b, the line parallel to L_1 passing through $L_1(T)$.

This system imposes $2(n-2)$ scalar constraints, which matches exactly the $2(n-2)$ degrees of freedom available in an $(n-2)$-fold configuration. Therefore, we obtain the following theorem. (See [2] for more detailed proof.)

THEOREM 4.2 ([2, Theorem 1]). *Every polynomial equation of degree n with real solutions can be solved by $(n-2)$-fold Origami.*

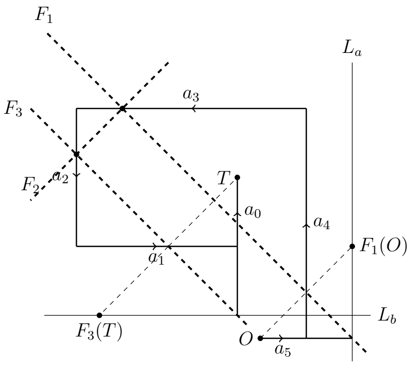


FIGURE 15. Lill's method realized by an $(n-2)$-fold construction.

5. Constructing Quadratic Complex Roots by Two-Fold Origami

All Origami construction methods discussed so far (RC-constructions, single-fold, two-fold, and multiple-fold Origami) operate in the Euclidean plane and therefore produce only real roots of polynomial equations. This naturally raises the question of whether Origami may also be used to obtain complex roots.

A graphical approach toward extending Lill's real-root method to the complex plane appears in late 19th-century French literature [3], and was recently revisited by Tabachnikov [16].

Building on this perspective, Nakai [14] formulated a unified geometric framework in which a polynomial with complex coefficients is represented as a polygonal path in the complex plane, and a complex root corresponds to a chain of similar triangles reflecting along its segments. To distinguish it from the real version discussed in Section 4, we refer to this extension as the *Lill's Method for Complex Roots* (LMCR).

Although Nakai proved that certain cubic equations can be solved theoretically using Origami together with LMCR, his work did not include explicit constructions. In this section, after illustrating LMCR through examples, we demonstrate that every quadratic equation with complex coefficients admits a fully explicit solution by a two-fold Origami construction.

Let $f(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial with complex coefficients. Define the points

$$C_0 := O, \quad C_1 := a_n, \quad \ldots, \quad C_k := a_n + a_{n-1} + \cdots + a_{n-k+1}, \quad C_{n+1} = a_n + \cdots + a_0.$$

Starting with $P_0 = C_0$, each subsequent point P_{k+1} is obtained by multiplying the directed segment $\overrightarrow{P_k C_{k+1}}$ by the complex factor $1 - z_0$. If this process yields $P_n = C_{n+1}$, then z_0 is a root of $f(z)$.

We do not prove the correctness of LMCR here, referring instead to Nakai [14]. We illustrate the mechanism using examples.

EXAMPLE 5.1. Consider the polynomial $f(z) = z^2 - (1+2i)z + (-1+i)$ whose roots are $1+i$ and i. We compute $C_0 = 0$, $C_1 = 1$, $C_2 = -2i$, and $C_3 = -1-i$. For the root $z_0 = 1+i$, multiplying $\overrightarrow{P_0C_1}$ by $1-(1+i) = -i$ yields $P_1 = -i$. Repeating with $\overrightarrow{P_1C_2}$ again multiplied by $-i$ gives $P_2 = -1-i = C_3$, confirming that $1+i$ is a root of $f(z)$. A similar computation verifies the root $z_0 = i$. The two LMCR traces are shown in Figures 16 and 17.

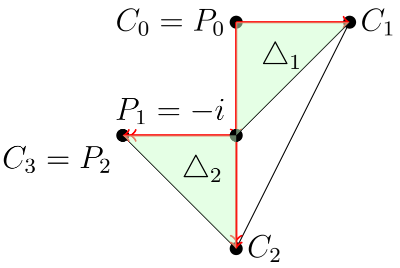


Figure 16. LMCR construction for the root $z = 1 + i$ of $f(z) = z^2 - (1 + 2i)z + (-1 + i)$

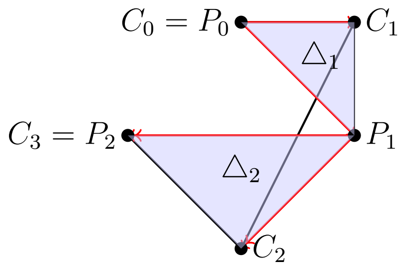


Figure 17. LMCR construction for the root $z = i$ of $f(z) = z^2 - (1 + 2i)z + (-1 + i)$

The construction typically produces a chain of similar triangles. To see this, consider the triangles $\triangle_1 := \triangle P_0P_1C_1$ and $\triangle_2 := \triangle P_1P_2C_2$ in Figure 16 (or Figure 17). Multiplication by $1 - z_0$ rotates by $\arg(1 - z_0)$ and scales by $|1 - z_0|$, so corresponding angles and side ratios agree:

$$\angle C_1P_0P_1 = \angle C_2P_1P_2, \quad \frac{|\overline{P_0P_1}|}{|\overline{P_0C_1}|} = \frac{|\overline{P_1P_2}|}{|\overline{P_1C_2}|}.$$

Thus $\triangle_1 \sim \triangle_2$. In fact, LMCR may be reformulated as the task of locating a sequence of similar triangles.

In most cases, finding such a sequence of similar triangles allows us to construct the roots. However, there are exceptional cases where the roots of certain polynomials do not correspond to a sequence of similar triangles. We discuss these cases in the following examples.

Example 5.2. If $f(z)$ has the root $z = 0$, then multiplying by $1 - 0 = 1$ gives $P_k = C_k$ for all k, so no triangle appears. Since the constant term vanishes, $C_{n+1} = C_n$, and the LMCR condition $P_n = C_{n+1}$ still holds. The root $z = 0$ may then be factored out and LMCR applied to the remaining polynomial.

If $f(z)$ has the root $z = 1$, then multiplying by $1 - 1 = 0$ forces $P_1 = O$ and hence $P_k = O$ for all $k \geq 1$. Because the sum of the coefficients of $f(z)$ equals zero, we again have $C_{n+1} = O$, so the LMCR condition $P_n = C_{n+1}$ is satisfied. The root $z = 1$ can then be factored out, and LMCR applied to the reduced polynomial.

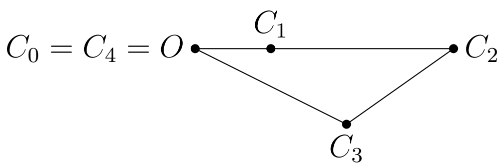


Figure 18. LMCR construction for the root $z = 1$ of $f(z) = z^3 + (1 + \sqrt{2})z^2 - (\sqrt{2} + i)z - 2 + i$

A further degenerate case arises when $\arg(1 - z_0) = 0$ or $\arg(1 - z_0) = \pi$, so multiplication by $1 - z_0$ produces no rotation and no similar triangles appear. This phenomenon appears precisely when the root z_0 is real. This issue can be resolved through a simple transformation.

Suppose $f(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ has a real root $r \in \mathbb{R}$. Define a new polynomial $f_0(z) = b_nz^n + b_{n-1}z^{n-1} + \cdots + b_1z + b_0$, where $b_k = i^{n-k}a_k$ for $k = 0, \ldots, n$.

Then $i^n f(z) = f_0(iz)$, so $f_0(z)$ has the root ir, which is non-real. Consequently, LMCR applied to f_0 yields a nondegenerate configuration with a valid chain of similar triangles.

If the polynomial $f(z)$ has non-real coefficients (after normalizing by dividing by a_n), we apply LMCR directly. If all coefficients are real, we instead apply LMCR to a rotated version of the polynomial, guaranteeing the appearance of non-real roots and thus a nondegenerate similar triangle configuration.

Once the sequence of n similar triangles is obtainable for a degree n polynomial, LMCR allows us to determine a root geometrically. The quadratic case is the first instance where a nontrivial similar-triangle configuration appears, and only two triangles are required. We now show that this configuration can be realized by a two-fold construction.

THEOREM 5.3. *Any quadratic equation with complex coefficients can be solved by two-fold Origami.*

Proof. As illustrated in Figure 19, let L_{k+1} denote the line extending the segment $\overline{C_k C_{k+1}}$. Construct the perpendicular bisector L_4 of the segment $\overline{C_1 C_2}$. When $C_1 = C_2$, we take $L_4 := L_1$. We then apply a two-fold axiom such as AL13a7bb or AL7a910a (see p. 13 in [2]) satisfying the following alignment conditions, as shown in Figure 20:

$$(1) \qquad \begin{cases} C_2 \in F_1, \\ F_2(C_0) \in F_1(L_4(L_1)), \\ F_2(C_0) \in L_4(F_1(L_3)), \\ F_1(L_4(F_2)) \perp L_{F_2(C_0),C_3}, \quad \text{where } L_{A,B} \text{ is a line containing } A \text{ and } B. \end{cases}$$

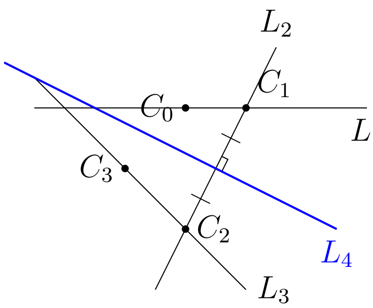


FIGURE 19. Initial construction: lines L_1, L_2, L_3 and perpendicular bisector L_4 for the example $z = i$ of $f(z) = z^2 - (1+2i)z + (-1+i)$

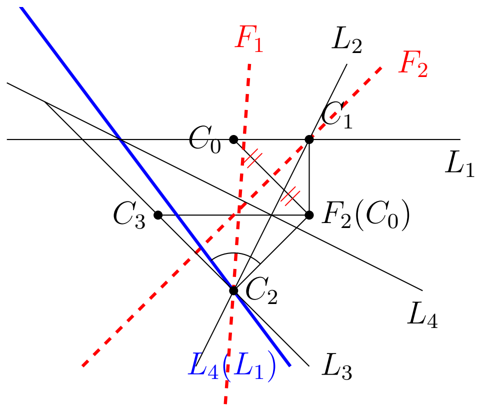


FIGURE 20. Two-fold construction using axiom AL13a7bb: fold lines F_1 and F_2 satisfying the alignment conditions

The key difficulty in this proof is to maintain a specific angle using only reflections and point alignments. The first three conditions ($C_2 \in F_1$, $F_2(C_0) \in F_1(L_4(L_1))$, and $F_2(C_0) \in L_4(F_1(L_3))$) ensure that the triangle $\triangle C_0 C_1 F_2(C_0)$ is first reflected across L_4 and then across F_1, resulting in a configuration congruent to $\triangle C_0' C_2 F_2(C_0)'$, where $C_0' = F_1(L_4(C_0))$ and $F_2(C_0)' = F_1(L_4(F_2(C_0)))$, as illustrated in Figure 21.

If the segment $\overline{C_0' F_2(C_0)'}$ is parallel to $\overline{C_3 F_2(C_0)}$, then the required pair of similar triangles has been obtained. This parallelism is guaranteed by the final condition $F_1(L_4(F_2)) \perp L_{F_2(C_0),C_3}$. Since F_2 is perpendicular to the line $L_{C_0,F_2(C_0)}$, the reflected line $F_1(L_4(F_2))$ is still perpendicular to $L_{C_0',F_2(C_0)'}$. Thus, the final condition $F_1(L_4(F_2)) \perp L_{F_2(C_0),C_3}$ implies that

$\overline{C_0'F_2(C_0)'}$ is parallel to $\overline{F_2(C_0)C_3}$. Consequently, the constructed point $F_2(C_0)$ coincides with P_1 in the LMCR construction, thereby determining the root. A direct comparison between Figure 21 and Figure 17 confirms that the two-fold construction produces the same root.

We have shown that a two-fold construction satisfying condition (1) produces a sequence of similar triangles. Conversely, assume we are given a sequence of similar triangles associated with a non-real root. We will construct two-fold lines that satisfy condition (1) (See Figure 17 through Figure 21). First, construct line L' as the perpendicular bisector of $\overline{C_1C_2}$. Then construct the fold line F_1 as the angle bisector of two lines $L_{L'(C_0),C_2}$ and L_{P_1,C_2}. Then the point $F_2(C_0)$ equals P_1, and thus two triangles $\triangle C_3C_2F_2(C_0)$ and $\triangle C_0C_1F_2(C_0)$ are similar, which implies condition (1) is satisfied. Since F_1 is the angle bisector of two lines $L_{L'(C_0),C_2}$ and L_{P_1,C_2}, there are two solutions for the same sequence of similar triangles. Therefore there is a 2 to 1 correspondence between two-fold lines satisfying condition (1) and the resulting LMCR sequence of similar triangles.

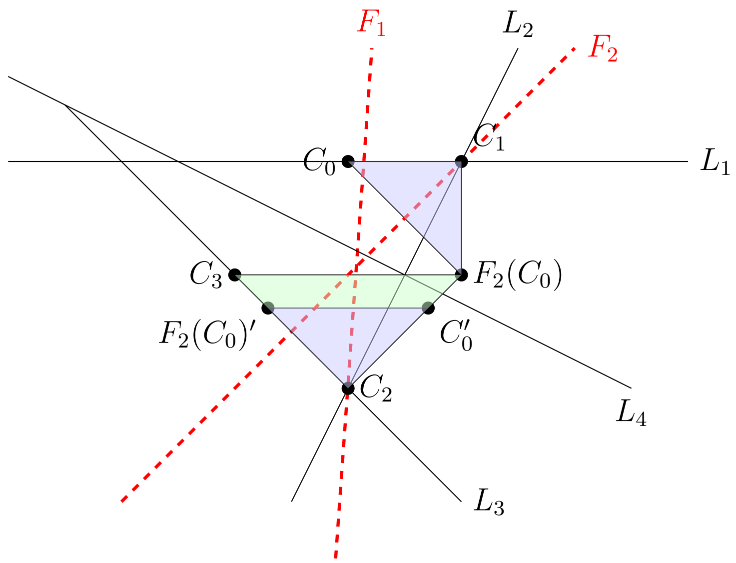


FIGURE 21. Reflection of $\triangle C_0C_1F_2(C_0)$ across L_4 and F_1 showing the similar triangle configuration

□

6. Conclusion

We have examined geometric constructions using Origami, demonstrating the progression from classical ruler-and-compass methods to powerful multiple fold techniques. The Huzita-Justin axioms provide a complete axiomatic foundation for single-fold Origami, with axioms O_1–O_5 exactly characterizing RC-constructible points and axiom O_6 enabling cubic equation solutions including angle trisection.

The introduction of multiple-fold axioms significantly extends constructive power. We showed that $(n-2)$-fold Origami can solve degree n polynomials using Lill's geometric method. Furthermore, by introducing Lill's Method for Complex Roots (LMCR), we presented an explicit construction method using two-fold Origami to solve all quadratic equations with complex coefficients via the axiom AL13a7bb.

It is natural to ask whether the LMCR construction can be generalized to higher-degree polynomials. As n increases, the number of alignment conditions grows, and representing specific angles through reflections and point-line incidences becomes increasingly intricate. Building on the explicit construction method developed here for quadratic equations, we plan to investigate similar approaches for cubic and higher-degree complex polynomials using multiple-fold Origami.

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